Eur. Phys. J. B (2012) 85: 349
DOI: 10.1140/epjb/e2012-30731-8

THE EUROPEAN PHYSICAL JOURNAL B

Regular Article

Universality aspects of the trimodal random-field Ising model

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Received 7 August 2012 / Received in final form 9 September 2012
Published online 18 October 2012 –

Abstract. We investigate the critical properties of the $d = 3$ random-field Ising model with an equal-weight trimodal distribution at zero temperature. By implementing suitable graph-theoretical algorithms, we compute large ensembles of ground states for several values of the disorder strength h and system sizes up to $N = 128^3$. Using a new approach based on the sample-to-sample fluctuations of the order parameter of the system and proper finite-size scaling techniques we estimate the critical disorder strength $h_c = 2.747(3)$ and the critical exponents of the correlation length $\nu = 1.34(6)$ and order parameter $\beta = 0.016(4)$. These estimates place the model into the universality class of the corresponding Gaussian random-field Ising model.

1 Introduction

The random-field Ising model (RFIM) [1–15] has been extensively studied due to its interest as a simple frustrated system, as well as its close connection to experiments [16–20]. Its beauty is that the mixture of random fields and the standard Ising model creates rich physics and leaves many still unanswered problems. The Hamiltonian describing the model is

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j - \sum_i h_i \sigma_i, \qquad (1)$$

where $\sigma_i = \pm 1$ are Ising spins, $J > 0$ is the nearest-neighbor's ferromagnetic interaction, and h_i are independent quenched random fields. Several field distributions have been considered in the literature, the most common being the Gaussian and bimodal distributions [18,21–24]. The existence of an ordered ferromagnetic phase for the RFIM, at low temperature and weak disorder, follows from the seminal discussion of Imry and Ma [1], when the space dimension is greater than two ($d > 2$) [21–26]. This has provided us with a general qualitative agreement on the sketch of the phase boundary, separating the ordered ferromagnetic phase from the high-temperature paramagnetic one. The phase-diagram line separates the two phases of the model and intersects the randomness axis at the critical value of the disorder strength h_c [27–35].

The criteria for determining the order of the low temperature phase transition and its dependence on the form of the field distribution have been discussed throughout the years [32–37]. Already from the work of Houghton and Khurana [38], the importance of the form of the distribution function in the determination of the critical properties of the RFIM has been emphasized. In fact, different results have been proposed for different field distributions, like the existence of a tricritical point at the strong disorder regime of the system, present only in the bimodal case [32–35,38]. Following the results of Houghton and Khurana [38], Mattis [39] reexamined the RFIM introducing a new type of distribution, the trimodal one, given by the form

$$\mathcal{P}(h_i) = p\delta(h_i) + (1/2)(1-p)[\delta(h_i - h) + \delta(h_i + h)], \qquad (2)$$

where h defines the disorder (field) strength and $p \in (0, 1)$. Clearly, for $p = 1$ one switches to the pure Ising model, whereas for $p = 0$ the well-known bimodal distribution is recovered. In general terms, the trimodal distribution (2) allows a physical interpretation as a diluted bimodal distribution, in which a fraction p of the spins are not exposed to the external field. Thus, it mimics the salient feature of the Gaussian distribution, for which a significant fraction of the spins are in weak external fields. Mattis suggested that for a particular case, $p = 1/3$, this may be considered as a good approximation to the Gaussian distribution [39]. This in turn indicated that the two models should be in the same universality class. Further studies along these lines, using mean-field and renormalization-group approaches, provided contradicting evidence for the critical aspects of the $p = 1/3$ model and also proposed several approximations of its phase diagram for a range of values of p [40–42].

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However, none of these predictions has been confirmed by numerical simulations up to now, thus remaining ambiguous, due to the approximate nature of the mean-field-type of methods used.

Currently, despite the huge efforts recorded in the literature, a clear picture of the model's critical behavior is still lacking. Although the view that the phase transition of the RFIM is of second-order, irrespective of $\mathcal{P}(h_i)$, is well established [43–46], the extremely small value of the exponent β continues to cast some doubts. Moreover, a rather strong debate exists with regards to the role of disorder: the available simulations are not able to settle the question of whether the critical exponents depend on the particular choice of the distribution for the random fields, analogously to the mean-field theory predictions [32–35]. Thus, the whole issue of universality, in terms of different field distributions and critical exponents, is under investigation [43–62].

In the present work we shed some light towards this direction by examining the critical features of the phase diagram of the trimodal RFIM for the case of $p = 1/3$. We provide numerical evidence that clarify the matching between the trimodal and Gaussian models and we give estimates for the critical exponents that compare very well to the most accurate ones in the corresponding literature of the RFIM. Our attempt benefits from: (i) the existence of robust computational methods of graph theory at zero temperature ($T = 0$), (ii) classical finite-size scaling (FSS) techniques, and (iii) a new scaling approach that involves the sample-to-sample fluctuations of the order parameter. In particular, sample-to-sample fluctuations and the relative issue of self-averaging have attracted much interest in the study of disordered systems [63]. Although it has been known for many years now that for (spin and regular) glasses there is no self-averaging in the ordered phase [64], for random ferromagnets such a behavior was first observed for the RFIM by Dayan et al. [65] and some years later for the random versions of the Ising and Ashkin-Teller models by Wiseman and Domany [66]. These latter authors suggested a FSS ansatz describing the absence of self-averaging and the universal fluctuations of random systems near critical points that was refined on a more rigorous basis by Aharony and Harris [67]. Ever since, the subject of breakdown of self-averaging is an important aspect of several theoretical and numerical investigations of disordered spin systems [68–81]. In fact, Efrat and Schwartz [82] showed that the property of lack of self-averaging may be turned into a useful tool that can provide an independent measure to distinguish the ordered and disordered phases of the system. In view of this increasing interest, we discuss here another successful alternative approach to the criticality of the RFIM via the sample-to-sample fluctuations of the order parameter at $T = 0$.

The rest of the paper is organized as follows: In the next Section we describe briefly the $T = 0$ numerical approach and we provide all the necessary details of our investigation. The relevant twofold FSS analysis of the numerical data, as well as the estimation procedure of the critical disorder strength h_c and the critical exponents ν and β are given in Section 3. The paper ends with a summary of our conclusions in Section 4.

2 Simulation protocol at zero temperature

As it is well known, the random field is a relevant perturbation at the pure fixed point, and the random-field fixed point is at $T = 0$ [18,21–25]. We can therefore determine the critical behavior, staying at $T = 0$ and crossing the phase boundary at $h = h_c$. This is a convenient approach because we can determine the ground states exactly using efficient optimization algorithms [83–107] through an existing mapping of the ground state to the maximum-flow optimization problem [108–110]. A clear advantage of this approach is the ability to simulate large system sizes and disorder ensembles in rather moderate computational times. We should underline here that, typical Monte Carlo simulations exhibit extremely slow dynamics in the low-temperature phase of these systems, due to the existence of many meta-stable states separated by barriers that grow with the system size [111,112]. Thus, even the most efficient numerical schemes at $T > 0$ are upper bounded by linear sizes of the order of $L_{max} \leq 32$. Further assets in the $T = 0$ approach are the absence of statistical errors and equilibration problems, which, on the contrary, are the two major drawbacks encountered in $T > 0$ simulations of systems with rough free-energy landscapes [18].

The application of maximum-flow algorithms to the RFIM is nowadays well established [100]. Nevertheless, we find useful to shortly review the algorithm. The network flow algorithm generally used to solve the RFIM is, because of its speed, the push-relabel (PR) algorithm of Tarjan and Goldberg [113,114]. For the interested reader, general proofs and theorems on the PR algorithm can be found in standard textbooks [109,110]. The version of the PR algorithm implemented in our study involves a modification proposed by Middleton and Fisher [43] that removes the source and sink nodes, reducing memory usage and also clarifying the physical connection [95–98]. The algorithm starts by assigning an excess x_i to each lattice site i, with $x_i = h_i$. Residual capacity variables r_{ij} between neighboring sites are initially set to J. A height variable u_i is then assigned to each node via a global update step. In this global update, the value of u_i at each site in the set $\mathcal{T} = \{j|x_j < 0\}$ of negative excess sites is set to zero. Sites with $x_i \geq 0$ have u_i set to the length of the shortest path, via edges with positive capacity, from i to $\mathcal{T}$.

The ground state is found by successively rearranging the excesses x_i, via *push* operations, and updating the heights, via *relabel* operations. When no more pushes or relabels are possible, a final global update determines the ground state, so that sites which are path connected by bonds with $r_{ij} > 0$ to $\mathcal{T}$ have $\sigma_i = -1$, while those which are disconnected from $\mathcal{T}$ have $\sigma_i = 1$. A push operation moves excess from a site i to a lower height neighbor j, if possible, that is, whenever $x_i > 0$, $r_{ij} > 0$, and $u_j = u_i - 1$. In a push, the working variables are modified according to $x_i \to x_i - \delta$, $x_j \to x_j + \delta$, $r_{ij} \to r_{ij} - \delta$, and $r_{ji} \to r_{ji} + \delta$,

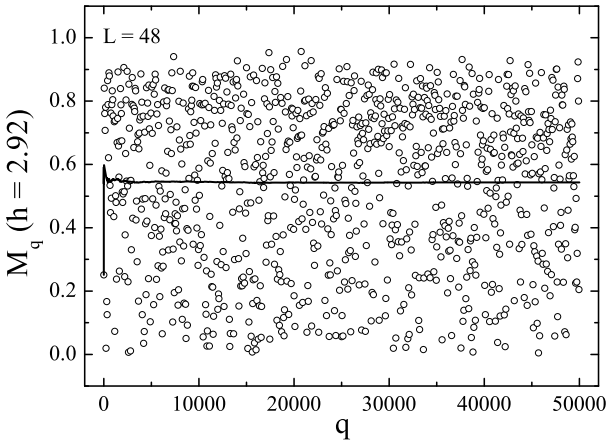

Fig. 1. Disorder distribution of the ground-state order parameter of the $p = 1/3$ trimodal RFIM for $L = 48$ and $h = 2.92$ using the *speed mode* representation, i.e., several data points have been removed to make the illustration clearer for the reader. The running average, estimated over the complete set of disorder samples (5×10^4), is shown by the solid line.

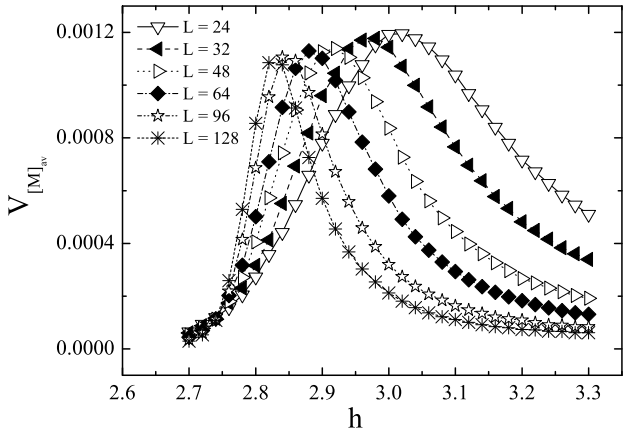

Fig. 2. Sample-to-sample fluctuations of the order parameter of the $p = 1/3$ trimodal RFIM as a function of the disorder strength for various lattice sizes. Lines are just guides to the eye.

with $\delta = \min(x_i, r_{ij})$. Push operations tend to move the positive excess towards sites in $\mathcal{T}$. When $x_i > 0$ and no push is possible, the site is relabelled, with u_i increased to $1 + \min_{\{j|r_{ij}>0\}} u_j$. In addition, if a set of highest sites $\mathcal{U}$ become isolated, with $u_i > u_j + 1$, for all $i \in \mathcal{U}$ and all $j \notin \mathcal{U}$, the height u_i for all $i \in \mathcal{U}$ is increased to its maximum value, N, as these sites will always be isolated from the negative excess nodes. Periodic global updates are often crucial to the practical speed of the algorithm [96–98]. Following the suggestions of Middleton et al. [43,95–98], we have also applied global updates here every N relabels, a practise found to be computationally optimum.

Using the above described version of the PR algorithm we performed large-scale simulations of the trimodal ($p = 1/3$) RFIM for a wide range of simulation parameters. In the first part, preliminary runs were executed, including also small systems sizes $N \leq 20^3$, in order to probe efficiently the critical h-regime of the model. In the second part, extensive simulations have been performed for lattice sizes $L \in \{24, 32, 48, 64, 96, 128\}$ and disorder strengths $h \in [2.7 - 3.3]$ with a step $\delta h = 0.02$. For each pair $(L,\ h)$ an extensive disorder averaging – denoted hereafter as $[\ldots]_{\rm av}$ – has been undertaken by sampling over $Q = 5 \times 10^4$ random-field realizations. Figure 1 presents evidence that the above number of random realizations is sufficient in order to obtain the true average behavior. In particular, we plot in this figure, for $L = 48$ and $h = 2.92$, the disorder distribution of the ground-state absolute value order parameter per spin, defined as $M_{\rm q} = \{|\sum_i \sigma_i| / N\}_{\rm q}$, where q defines a particular random realization of the external magnetic field h_i and runs over the ensemble of realizations as $q = 1, \ldots, Q$. We have chosen on purpose the random-field value $h = 2.92$, which as will be seen below, corresponds to the pseudo-critical disorder strength for the lattice size $L = 48$. The interesting point is that, for these L-dependent pseudo-critical values of h, one expects sample-to-sample fluctuations to be maximized and this becomes clear in Figure 1 through the large deviation of the $M_{\rm q}$ values. Figure 1 includes also the corresponding running average over the data, illustrated by the solid line. This is a series of averages of different subsets of the full data set – each of which is the average of the corresponding subset of a larger set of data points, over the samples for the simulated ensemble of 5×10^4 disorder realizations.

3 Finite-size scaling analysis and critical exponents

We start our analysis of the $p = 1/3$ trimodal RFIM with Figure 2, where we plot the sample-to-sample fluctuations of the disorder-averaged order parameter $V_{[M]_{\rm av}} = \sqrt{([M^2]_{\rm av} - [M]^2_{\rm av})/(Q-1)}$, where $[M]_{\rm av} = \left[\sum_q M_{\rm q}\right]/Q$ as usual, as a function of the disorder strength h for $L = 24$–128. It is clear that for every lattice size L, these fluctuations appear to have a maximum value at a certain value of h, denoted hereafter as $h^*_{\rm L}$, that may be considered in the following as a suitable pseudo-critical disorder strength. By fitting the data points around the maximum first to a Gaussian, and subsequently to a fourth-order polynomial, we have extracted the values of the peak-locations ($h^*_{\rm L}$) by taking the mean value via the two fitting functions, as well as the corresponding error bars. Using now these values for $h^*_{\rm L}$ we consider in the main panel of Figure 3 a power-law fitting attempt of the form

$$h^*_{\rm L} = h_{\rm c}^{(\rm eff)} + bL^{-1/\nu^{(\rm eff)}}. \qquad (3)$$

This fitting produces effective estimates for the critical disorder strength and the correlation length's exponent. In Figure 3 we show the fitting for the lattice range $L = L_{\rm min} - 128$, where $L_{\rm min} = 24$. By applying the same procedure for $L_{\rm min} = 32$ and 48, we obtained a set of three effective estimates for both quantities and

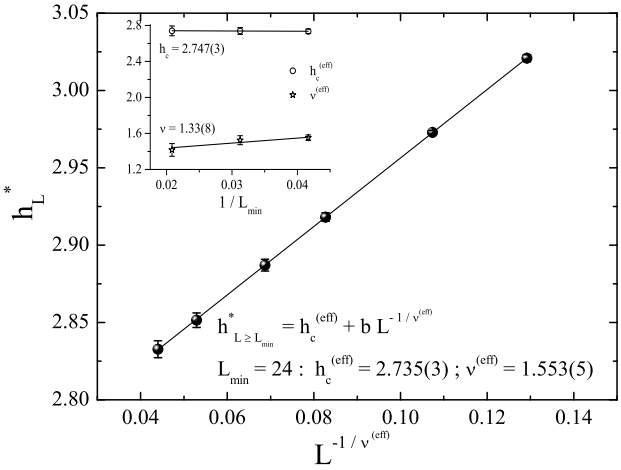

Fig. 3. FSS of the pseudo-critical disorder strengths $h_{\rm L}^*$ for $L \geq L_{\rm min} = 24$ (main panel). Infinite-limit size extrapolation of the effective estimates $\nu^{(\rm eff)}$ and $h_{\rm c}^{(\rm eff)}$ obtained via the fitting procedure of the main panel in the range $L_{\rm min} - 128$, where $L_{\rm min} = 24$, 32, and 48 (inset).

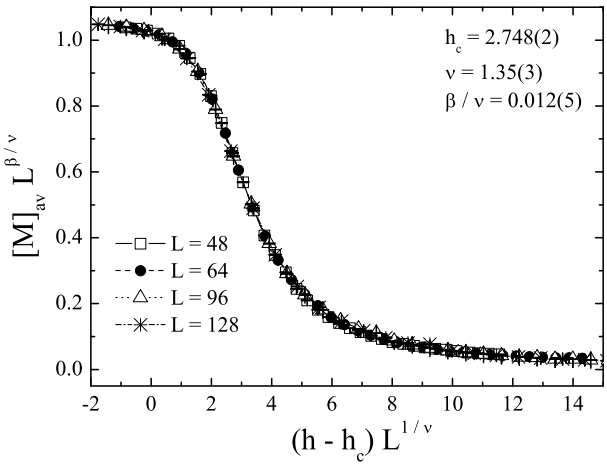

Fig. 4. Scaling plot of the disorder-averaged magnetization using the data-collapse approach described in the text.

we plot their infinite lattice-size extrapolation in the inset of Figure 3. The final values of the critical disorder strength and the correlation length's exponent, as given by a simple linear fitting to $L \to \infty$, are $h_{\rm c} = 2.747(3)$ and $\nu = 1.33(8)$. This latter value for the critical exponent of the correlation length ν is within errors inside the range $\nu = 1.32(7) - 1.37(9)$ proposed for the Gaussian RFIM by Middleton and Fisher (MF) [43] and Hartmann and Young (HY) [90].

We note here that our suggestion of choosing these newly defined pseudo-critical disorder strengths $h_{\rm L}^*$ as a proper measure for performing FSS closely follows the analogous considerations of HY for the case of the Gaussian RFIM [90]. These authors considered pseudo-critical disorder strengths at the values of h at which a specific-heat-like quantity obtained by numerically differentiating the bond energy with respect to h attains its maximum. It appears that, this method is capable of producing very accurate estimates for both the critical disorder strength and also the correlation length's exponent, assuming that its behavior follows the observed shift behavior of our pseudo-critical disorder strengths $h_{\rm L}^*$. It is well known from the general scaling theory that, even for simple models, the equality between the correlation length's exponent and the shift exponent is not a necessary consequence of scaling [115]. Of course, it is a general practice to assume that the correlation length's behavior can be deduced by the shift of appropriate thermodynamic functions. Moreover, MF using similar reasoning on the Gaussian RFIM, characterized the distribution of the order parameter by the average over samples of the square of the magnetization per spin and the root-mean-square sample-to-sample variations of the square of the magnetization [43]. They identified a similar behavior to that of Figure 2, i.e., with increasing L, the peak magnitude of this quantity moved its location to smaller values of h, defining another relevant pseudo-critical disorder strength. However, MF were interested on the scaling behavior of the height of these peaks [43].

A further verification of the above estimates, together with an independent calculation of the magnetic exponent ratio β/ν, comes from a rather more common scaling procedure. In Figure 4 we show a collapse of the magnetization data, via the scaling relation

$$[M]_{\rm av} = L^{-\beta/\nu}[\tilde{M}]_{\rm av}\left[(h - h_{\rm c})L^{1/\nu}\right], \qquad (4)$$

where the function $\tilde{M}$ is a universal function of the scaling variable $(h - h_{\rm c})L^{1/\nu}$ which is, according to the theory of FSS, independent of the microscopic parameters of the system [115,116]. For the fitting procedure we have used *autoScale*, a program that performs a FSS analysis for given sets of simulated data [116]. The program implements a general scaling assumption and optimizes an initial set of scaling parameters that enforce a data collapse of the different sets. The optimum data collapse, found by the minimization procedure of the scaling parameters via the downhill simplex algorithm [117], emerged for $h = 2.7$–3.2 and $L = 48$–128, and is shown in Figure 4. The resulting values of the scaling parameters, $h_{\rm c} = 2.748(2)$ and $\nu = 1.35(3)$, are in excellent agreement with those of Figure 3, obtained via the FSS of the proposed pseudo-critical disorder strengths $h_{\rm L}^*$. Moreover, the value of the magnetic exponent ratio $\beta/\nu = 0.012(5)$ indicates, through $\nu = 1.35(3)$, that $\beta = 0.016(4)$, which compares very well to the most accurate estimation in the literature of the Gaussian RFIM, i.e., $\beta = 0.017(5)$ [43].

Closing, we provide in Figure 5 a schematic representation of some random ground-state spin configurations of the system. In particular, we consider a system with linear size $L = 24$ and 3 values of the disorder strength around the critical $h_{\rm c}$-point, namely $h = 2.6$, 3.02, and 3.2. In Figure 5 blue cones refer to ($\downarrow$) spins, whereas red cones to ($\uparrow$) spins. According to the standard picture, when $h < h_{\rm c}$, the ferromagnetic interaction between nearest neighbors dominates and the spins take on a mean value

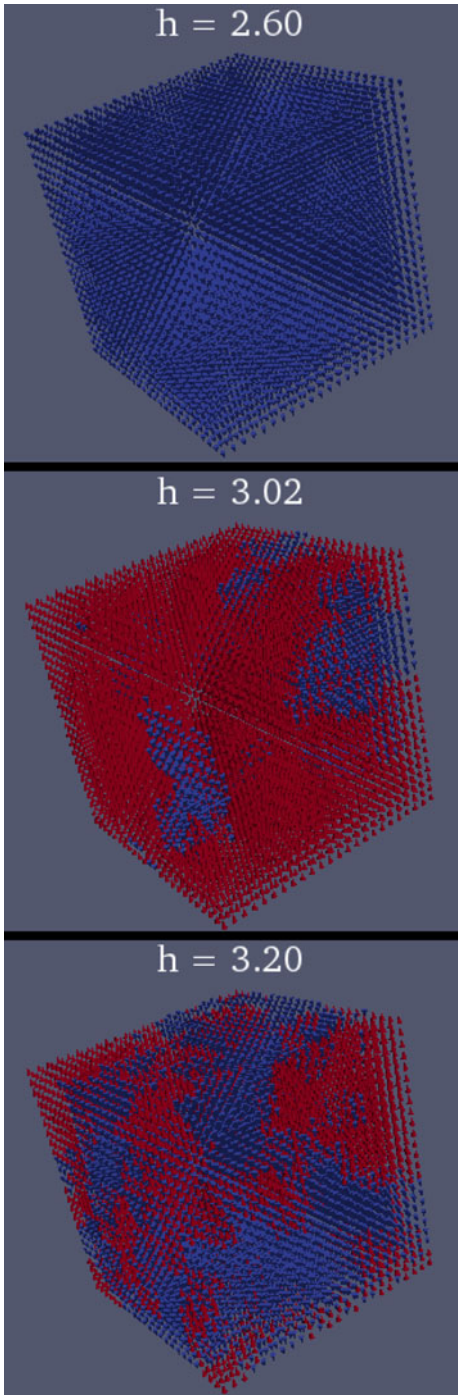

Fig. 5. (Color online) Ground-state configurations of a lattice size $L = 24$ for 3 values of h, as indicated.

$\left|\frac{1}{N}\sum_i \sigma_i\right| \neq 0$, as $N \to \infty$. With increasing field strength and as we approach the paramagnetic regime, $h > h_c$, randomness dominates through the formation of domains of different spin orientations, leading to $\left|\frac{1}{N}\sum_i \sigma_i\right| = 0$, as $N \to \infty$. Let us note here that, for the lattice size $L = 24$ used in the illustration of Figure 5, the corresponding value of the pseudo-critical disorder strength is $h^*_{24} \cong 3.02$.

4 Conclusions

To summarize, we have investigated the ground-state criticality of the $d = 3$ trimodal RFIM for a particular value of p, namely $p = 1/3$, for which a matching to the corresponding Gaussian model has been proposed [39]. In particular, we have estimated the critical disorder strength $h_c = 2.747(3)$ and the critical exponents $\nu = 1.34(6)$ and $\beta = 0.016(4)$ of the correlation length and order parameter, respectively. These values, obtained through different scaling techniques, compare well enough to the most accurate estimates of the literature, placing the current model into the universality class of the Gaussian RFIM. Our effort became feasible through the implementation of a modified version of the PR algorithm [43] that enabled us to simulate very large system sizes, up to 128^3 spins, and disorder ensembles of the order of 5×10^4, for several values of the random-field strength. Clearly, such a computational task goes beyond the limits of any kind of $T > 0$ Monte Carlo scheme. An interesting aspect of our analysis has been the illustration that quantities related to the sample-to-sample fluctuations of the order parameter of the system constitute a useful alternative to criticality.

N.G.F. has been partly supported by MICINN, Spain, through Research Contract No. FIS2009-12648-C03. P.E.T. is grateful for financial support by the Austrian Science Foundation within the SFB ViCoM (Grant F41). I.G. acknowledges financial support by *Marie Curie ITN-COMPLOIDS* (Grant Agreement No. 234810).

References

1. Y. Imry, S.-K. Ma, Phys. Rev. Lett. **35**, 1399 (1975)
2. A. Aharony, Y. Imry, S.-K. Ma, Phys. Rev. Lett. **37**, 1364 (1976)
3. A.P. Young, J. Phys.: Condens. Matter **10**, L257 (1977)
4. S. Fishman, A. Aharony, J. Phys.: Condens. Matter **12**, L729 (1979)
5. G. Parisi, N. Sourlas, Phys. Rev. Lett. **43**, 744 (1979)
6. J.L. Cardy, Phys. Rev. B **29**, 505 (1984)
7. J.Z. Imbrie, Phys. Rev. Lett. **53**, 1747 (1984)
8. M. Schwartz, A. Soffer, Phys. Rev. Lett. **55**, 2499 (1985)
9. M. Schwartz, A. Soffer, Phys. Rev. B **33**, 2059 (1986)
10. M. Schwartz, J. Phys.: Condens. Matter **18**, 135 (1985)
11. M. Schwartz, M. Gofman, T. Nattermann, Physica A **178**, 6 (1991)
12. M. Schwartz, Europhys. Lett. **15**, 777 (1994)
13. M. Gofman, J. Adler, A. Aharony, A.B. Harris, M. Schwartz, Phys. Rev. Lett. **71**, 1569 (1993)
14. J. Esser, U. Nowak, Phys. Rev. B **55**, 5866 (1997)
15. W.C. Barber, D.P. Belanger, J. Magn. Magn. Mater. **226**, 545 (2001)
16. D.P. Belanger, A.P. Young, J. Magn. Magn. Mater. **100**, 272 (1991)
17. H. Rieger, in *Annual Reviews of Computational Physics II*, edited by D. Stauffer (World Scientific, Singapore 1995), pp. 295–341

18. See, e.g., the articles by D.P. Belanger, T. Nattermann in *Spin Glasses and Random Fields*, edited by A.P. Young (World Scientific, 1998)
19. D.P. Belanger, A.R. King, V. Jaccarino, J.L. Cardy, Phys. Rev. B **28**, 2522 (1983)
20. D.P. Belanger, Z. Slanič, J. Magn. Magn. Mater. **186**, 65 (1998)
21. J. Villain, Phys. Rev. Lett. **52**, 1543 (1984)
22. J. Villain, J. Phys. **46**, 1843 (1985)
23. A.J. Bray, M.A. Moore, J. Phys.: Condens. Matter **18**, L927 (1985)
24. D.S Fisher, Phys. Rev. Lett. **56**, 416 (1986)
25. A.N. Berker, S.R. McKay, Phys. Rev. B **33**, 4712 (1986)
26. J. Bricmont, A. Kupiainen, Phys. Rev. Lett. **59**, 1829 (1987)
27. M.E.J. Newman, B.W. Roberts, G.T. Barkema, J.P. Sethna, Phys. Rev. B **48**, 16533 (1993)
28. J. Machta, M.E.J. Newman, L.B. Chayes, Phys. Rev. E **62**, 8782 (2000)
29. M.E.J. Newman, G.T. Barkema, Phys. Rev. E **53**, 393 (1996)
30. M. Itakura, Phys. Rev. B **64**, 012415 (2001)
31. N.G. Fytas, A. Malakis, Eur. Phys. J. B **61**, 111 (2008)
32. A. Aharony, Phys. Rev. B **18**, 3318 (1978)
33. A. Aharony, Phys. Rev. B **18**, 3328 (1978)
34. T. Schneider, E. Pytte, Phys. Rev. B **15**, 1519 (1977)
35. D. Andelman, Phys. Rev. B **27**, 3079 (1983)
36. S. Galam, J.L. Birman, Phys. Rev. B **28**, 5322 (1983)
37. V.K. Saxena, Phys. Rev. B **30**, 4034 (1984)
38. A. Houghton, A. Khurana, F.J. Seco. Phys. Rev. Lett. **55**, 856 (1985)
39. D.C. Mattis, Phys. Rev. Lett. **55**, 3009 (1985)
40. M. Kaufman, P.E. Klunzinger, A. Khurana, Phys. Rev. B **34**, 4766 (1986)
41. R.M. Sebastianes, V.K. Saxena, Phys. Rev. B **35**, 2058 (1987)
42. A.S. de Arruda, W. Figueiredo, R.M. Sebastianes, V.K. Saxena, Phys. Rev. B **39**, 4409 (1989)
43. A.A. Middleton, D.S. Fisher, Phys. Rev. B **65**, 134411 (2002)
44. R.L.C. Vink, T. Fischer, K. Binder, Phys. Rev. E **82**, 051134 (2010)
45. L.A. Fernández, V. Martín-Mayor, D. Yllanes, Phys. Rev. B **84**, 100408(R) (2011)
46. N.G. Fytas, A. Malakis, K. Eftaxias, J. Stat. Mech.: Theor. Exp. P03015 (2008)
47. H. Rieger, A.P. Young, J. Phys. A: Math. Gen. **26**, 5279 (1993)
48. H. Rieger, Phys. Rev. B **52**, 6659 (1995)
49. A. Falicov, A.N. Berker, S.R. McKay, Phys. Rev. B **51**, 8266 (1995)
50. M.R. Swift, A.J. Bray, A. Martian, M. Cieplak, J.R. Banavar, Europhys. Lett. **38**, 273 (1997)
51. J.-C. Anglés d'Auriac, N. Sourlas, Europhys. Lett. **39**, 473 (1997)
52. N. Sourlas, Comput. Phys. Commun. **121**, 183 (1999)
53. U. Nowak, K.D. Usadel, J. Esser, Physica A **250**, 1 (1998)
54. P.M. Duxbury, J.H. Meinke, Phys. Rev. E **64**, 036112 (2001)
55. L. Hernández, H. Ceva, Physica A **387**, 2793 (2008)
56. N. Crokidakis, F.D. Nobre, J. Phys.: Condens. Matter **20**, 145211 (2008)
57. O.R. Salmon, N. Crokidakis, F.D. Nobre, J. Phys.: Condens. Matter **21**, 056005 (2009)
58. I.A. Hadjiagapiou, Physica A **390**, 2229 (2011)
59. I.A. Hadjiagapiou, Physica A **390**, 3204 (2011)
60. I.A. Hadjiagapiou, Physica A **391**, 3541 (2012)
61. Ü. Akinci, Y. Yüksel, H. Polat, Phys. Rev. E **83**, 061103 (2011)
62. M. Tissier, G. Tarjus, Phys. Rev. Lett. **107**, 041601 (2011)
63. R. Brout, Phys. Rev. **115**, 824 (1959)
64. K. Binder, A.P. Young, Rev. Mod. Phys. **58**, 837 (1986)
65. I. Dayan, M. Schwartz, A.P. Young, J. Phys.: Condens. Matter **26**, 3093 (1993)
66. S. Wiseman, E. Domany, Phys. Rev. E **52**, 3469 (1995)
67. A. Aharony, A.B. Harris, Phys. Rev. Lett. **77**, 3700 (1996)
68. K. Eichhorn, K. Binder, J. Phys.: Condens. Matter **8**, 5209 (1996)
69. F. Pázmándi, R. Scalettar, G.T. Zimányi, Phys. Rev. Lett. **79**, 5130 (1997)
70. S. Wiseman, E. Domany, Phys. Rev. Lett. **81**, 22 (1998)
71. H.G. Ballesteros, L.A. Fernández, V. Martín-Mayor, A. Muñoz Sudupe, G. Parisi, J.J. Ruiz-Lorenzo, Phys. Rev. B **58**, 2740 (1998)
72. Y. Tomita, Y. Okabe, Phys. Rev. E **64**, 036114 (2001)
73. G. Parisi, N. Sourlas, Phys. Rev. Lett. **89**, 257204 (2002)
74. G. Parisi, M. Picco, N. Sourlas, Europhys. Lett. **66** 465 (2004)
75. P.E. Berche, C. Chatelain, B. Berche, W. Janke, Eur. Phys. J. B **38**, 463 (2004)
76. C. Monthus, T. Garel, Eur. Phys. J. B **48**, 393 (2005)
77. A. Malakis, N.G. Fytas, Phys. Rev. E **73**, 016109 (2006)
78. Y. Wu, J. Machta, Phys. Rev. Lett. **95** 137208 (2005)
79. Y. Wu, J. Machta, Phys. Rev. B **74**, 064418 (2006)
80. A. Gordillo-Guerrero, J.J. Ruiz-Lorenzo, J. Stat. Mech.: Theor. Exp., P0601 (2007)
81. N.G. Fytas, A. Malakis, Phys. Rev. E **81**, 041109 (2010)
82. A. Efrat, M. Schwartz, `arXiv:cond-mat/0608435`
83. A.T. Ogielski, Phys. Rev. Lett. **57**, 1251 (1986)
84. A.K. Hartmann, K.D. Usadel, Physica A **214**, 141 (1995)
85. A.K. Hartmann, Physica A **248**, 1 (1998)
86. S. Bastea, P.M. Duxbury, Phys. Rev. E **58**, 4261 (1998)
87. S. Bastea, Phys. Rev. E **58**, 7978 (1998)
88. S. Bastea, P.M. Duxbury, Phys. Rev. E **60**, 4941 (1999)
89. A.K. Hartmann, U. Nowak, Eur. Phys. J. B **7**, 105 (1999)
90. A.K. Hartmann, A.P. Young, Phys. Rev. B **64**, 180404 (2001)
91. A.K. Hartmann, Phys. Rev. B **65**, 174427 (2002)
92. E.T. Seppälä, M.J. Alava, Phys. Rev. E **63**, 066109 (2001)
93. E.T. Seppälä, M.J. Alava, P.M. Duxbury, Phys. Rev. E **63**, 066110 (2001)
94. E.T. Seppälä, A.M. Pulkkinen, M.J. Alava, Phys. Rev. B **66**, 144403 (2002)
95. A.A. Middleton, Phys. Rev. Lett. **88**, 017202 (2002)
96. A.A. Middleton, `arXiv:cond-mat/0208182`
97. J.H. Meinke, A.A. Middleton, `arXiv:cond-mat/0502471`
98. D.C. Hambrick, J.H. Meinke, A.A. Middleton, `arXiv:cond-mat/0501269`
99. I. Dukovski, J. Machta, Phys. Rev. B **67**, 014413 (2003)
100. M.J. Alava, P.M. Duxbury, C.F. Moukarzel, H. Rieger, in *Phase Transitions and Critical Phenomena*, edited by C. Domb, J.L. Lebowitz (Academic Press, San Diego, 2001), Vol. 18
101. M. Zumsande, M.J. Alava, A.K. Hartmann, J. Stat. Mech.: Theor. Exp., P02012 (2008)

102. G.P. Shrivastav, S. Krishnamoorthy, V. Banerjee, S. Puri, Europhys. Lett. **96**, 36003 (2011)
103. B. Ahrens, A.K. Hartmann, Phys. Rev. B **83**, 014205 (2011)
104. B. Ahrens, A.K. Hartmann, Phys. Rev. B **84**, 144202 (2011)
105. B. Ahrens, A.K. Hartmann, Phys. Rev. B **85**, 224421 (2012)
106. J.D. Stevenson, M. Weigel, Europhys. Lett. **95**, 40001 (2011)
107. J.D. Stevenson, M. Weigel, Comput. Phys. Commun. **182**, 1879 (2011)
108. J.-C. Anglés d'Auriac, M. Preissmann, R. Rammal, J. Phys. Lett. **46**, L173 (1985)
109. T.H. Cormen, C.E. Leiserson, R.L. Rivest, *Introduction To Algorithms* (MIT Press, Cambridge, MA, 1990)
110. C.H. Papadimitriou, *Computational Complexity* (Addison-Wesley, Reading, MA, 1994)
111. A.K. Hartmann, H. Rieger, *Optimization Algorithms in Physics* (Wiley-VCH, Berlin, 2004)
112. A.K. Hartmann, M. Weigt, *Phase Transitions in Combinatorial Optimization Problems* (Wiley-VCH, Berlin, 2005)
113. A.V. Goldberg, R.E. Tarjan, J. Assoc. Comput. Mach. **35**, 921 (1988)
114. B.V. Cherkassky, A.V. Goldberg, Algorithmica **19**, 390 (1997)
115. M.N. Barber, in *Phase Transitions and Critical Phenomena*, edited by C. Domb, J.L. Lebowitz (Academic, NY, 1983)
116. O. Melchert, `arXiv:0910.5403`
117. J.A. Nelder, R. Mead, Comput. J. **7**, 308 (1965)