Klassische Teilchentheorie

$$H(x,p) = \frac{p^2}{2m} + V(x) \quad , \quad T(p) = \frac{p^2}{2m}$$

$$\boxed{\begin{aligned} \frac{dx(t)}{dt} &= \left.\frac{\partial H(x,p)}{\partial p}\right|_{\substack{x=x(t)\\ p=p(t)}} = \frac{p(t)}{m} \\ \frac{dp(t)}{dt} &= -\left.\frac{\partial H(x,p)}{\partial x}\right|_{\substack{x=x(t)\\ p=p(t)}} = -\left.\frac{dV(x)}{dx}\right|_{x=x(t)} \\ &= K(x(t)) \end{aligned}}$$

AB : $x(t_0)$, $p(t_0)$

$$H(x(t), p(t)) = H(x(t_0), p(t_0)) = E$$

$$E(t) = T(t) + V(t) = T(t_0) + V(t_0) = E(t_0)$$

$t \geq t_0$

mit $T(p(t)) \equiv T(t)$ etc.

Energieerhaltung

Beispiel: Linearer harmonischer Oszillator

$$V(x) = \frac{m\omega_0^2}{2} x^2 , \quad K(x) = -f x$$

$$\text{mit} \quad \omega_0 = \sqrt{\frac{f}{m}}$$

$$\boxed{\begin{aligned} \frac{dx(t)}{dt} &= \frac{p(t)}{m} \\ \frac{dp(t)}{dt} &= -m\omega_0^2 x(t) \end{aligned}}$$

AB: ($t_0 = 0$ gewählt)

$x(0)$, $p(0)$

$\Longrightarrow$

$$x(t) = x(0) \cos\omega_0 t + \frac{p(0)}{m\omega_0} \sin\omega_0 t$$

$$p(t) = p(0) \cos\omega_0 t - m\omega_0 x(0) \sin\omega_0 t$$

für $t \geq 0$

Klassische Feldtheorie

$$\boxed{\left\{-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right\}\psi(x,t) = i\hbar\frac{\partial\psi(x,t)}{\partial t}}$$

AB: $\psi(x,t_0)$ $\quad\psi$ Materiefeldstärke

$$\boxed{\rho(x,t) = |\psi(x,t)|^2 \quad \text{Materiedichte (Massendichte)}}$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi = i\hbar\frac{\partial\psi}{\partial t} \quad \Big|\cdot\frac{\psi^*}{i\hbar}$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi^*}{\partial x^2} + V\psi^* = -i\hbar\frac{\partial\psi^*}{\partial t} \quad \Big|\cdot\left(-\frac{\psi}{i\hbar}\right)$$

$$-\frac{\hbar}{2mi}\left(\psi^*\frac{\partial^2\psi}{\partial x^2} - \psi\frac{\partial^2\psi^*}{\partial x^2}\right)$$

$$= \psi^*\frac{\partial\psi}{\partial t} + \psi\frac{\partial\psi^*}{\partial t}$$

$$-\frac{\hbar}{2mi}\frac{\partial}{\partial x}\left(\psi^*\frac{\partial\psi}{\partial x} - \psi\frac{\partial\psi^*}{\partial x}\right) = \frac{\partial}{\partial t}\underbrace{(\psi^*\psi)}_{\rho}$$

$$\boxed{\vec{j}(x,t) = \frac{\hbar}{2mi}\left(\psi^*\frac{\partial\psi}{\partial x} - \psi\frac{\partial\psi^*}{\partial x}\right)\vec{e}_x}$$

$\psi(x,t)$

Materiestromdichte (Massenstromdichte)

$$\boxed{\operatorname{div}\vec{j} + \frac{\partial\rho}{\partial t} = 0}$$

Erhaltung der Masse (differentiell)

$$\frac{\partial}{\partial t}\rho = -\frac{\hbar}{2mi}\frac{\partial}{\partial x}\left(\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x}\right)$$

$$\frac{d}{dt}\int_{\mathbb{R}} dx\, \rho(x,t) = -\frac{\hbar}{2mi}\left(\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x}\right)\Bigg|_{-\infty}^{+\infty}$$

$= 0$ falls keine Quellen bzw. Senken im Unendlichen

$\Longrightarrow$

$$\boxed{\begin{aligned}\int_{\mathbb{R}} dx\, \rho(x,t) = M(t) = \int_{\mathbb{R}} dx\, \rho(x,t_0) &= M(t_0)\\ &= M = \text{konst.}\end{aligned}}$$

Erhaltung der Gesamtmasse

Bemerkung (s. später):

$$E(t) = \int_{\mathbb{R}} dx\, \underbrace{\psi^*(x,t)\left\{-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right\}\psi(x,t)}_{u(x,t)\ \text{Energiedichte}}$$

$E(t) = E(t_0)$, $t \geq t_0$, Energieerhaltung